JOURNAL OF DIFFERENTIAL EQUATIONS 45, 182–190 (1982)

On Splitting the Area under Curves II

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Received February 16, 1981; revised August 11, 1981

This paper continues the author's work [3, S. Minsker, *J. Differential Equations*, **26**, No. 3 (1977), 443–457.] on an area-splitting problem leading to the functional differential equation $a'(a(x)) = a(x)/x$. This equation is dealt with by transforming it into the linear equation $\psi'(x) = \psi(x+c)$, for which positive solutions on $(-\infty, -c)$ are sought.

1. INTRODUCTION

This paper continues the study of the second area-splitting problem presented in [3], in which we seek all continuous, strictly monotone functions f: $[0, r) \to R$ such that, on any interval $[0, x]$ with $0 < x < r$, the area under the graph of f is divided in a certain fixed ratio $\alpha/(1-\alpha)$ (independently of the choice of interval) at the place where f takes its average value on $[0, x]$. We saw in [3] that this problem is essentially equivalent to finding all functions $a \in C^1((0, r))$ with $0 < a(x) < x$ which satisfy the functional differential equation

$$a'(a(x)) = a(x)/x \qquad \text{for} \quad x \in (0, r). \tag{$*$}$$

In Section 2 of the present paper, we use an idea of Barba [1] to transform Eq. (*) into the linear differential-difference equation

$$\psi'(x) = \psi(x+c) \qquad \text{for} \quad x \in (-\infty, -c). \tag{$**$}$$

The quantities α and c are related via an initial condition. Both the method of steps and of forming linear combinations of exponential functions are employed (respectively in Sections 3 and 4) to generate positive solutions of (**), which in turn yield solutions of (*) and of the original problem. Results of Schürer [5] are used in Section 5 to completely resolve the case $r = \infty$, and to ensure the generality of our approach in Section 4. Many of the open questions raised in [3, Sect. 5] are resolved, including the existence of $1/e$-semiwombat functions (Theorem 10).

0022-0396/82/080182-09$02.00/0

2. The "Semiwombat" Problem

The problem is stated in [3, Sect. 3]. At this point, the reader should become familiar with the problem and its notation, and with the results stated in Lemma 7 through Theorem 14 of [3].

Let f be an α-semiwombat function on $[0, r)$ and let a be its associated average-value function, as in Eq. (16) of [3]. We know that $a \in C^1((0, r))$ with $0 < a(x) < x$, $a'(x) > 0$, and

$$a'(a(x)) = a(x)/x \qquad \text{for} \quad x \in (0, r). \tag{*}$$

Since a is strictly increasing, we shall denote $\lim_{x \to r} a(x)$ by $a(r)$.

LEMMA 1. *If* $r < \infty$, *then* $0 < a(r) < r$. *If* $r = \infty$, *then* $a(r) = \infty$.

Proof. If $r < \infty$, it is obvious that $0 < a(r) \leqslant r$. If $a(r) = r$, the mean-value theorem would imply $a'(r_1) = 1$ for some $r_1 \in (0, r)$, contradicting the fact that (*) implies $a' < 1$ on the range of a. If $r = \infty$ but $a(r)$ were finite, letting $x \to \infty$ in (*) would yield $a'(a(r)) = 0$, a contradiction.

We now proceed along the lines of Barba [1] to transform (*) into a linear equation. Let $g\colon (0, a(r)) \to (0, r)$ be the inverse of a. Then $g \in C^1((0, a(r)))$ with $g(x) > x$ and $g'(x) > 0$ for $x \in (0, a(r))$, and (*) implies

$$g'(x)\, g(x) = g(g(x)) \qquad \text{for} \quad x \in (0, a(a(r))). \tag{1}$$

Temporarily assume that $r < \infty$, and define

$$\phi(x) = \int_{a(r)}^{x} \frac{1}{g(t)}\, dt \qquad \text{for} \quad x \in (0, a(r)). \tag{2}$$

Then $\phi \in C^2((0, a(r)))$ with $1/x > \phi'(x) > 0$ and $\phi''(x) < 0$ for $x \in (0, a(r))$, and there is a constant c such that

$$\phi(g(x)) = \phi(x) + c \qquad \text{for} \quad x \in (0, a(a(r))). \tag{3}$$

(This is a consequence of (1).)

LEMMA 2. $c = -\alpha \log \alpha$. *Hence* $0 < c \leqslant 1/e$. *Also*, $\lim_{x \to 0} \phi(x) = -\infty$.

Proof. From (2) and (3),

$$c = \int_{x}^{g(x)} \frac{1}{g(t)}\, dt = \int_{g(x)}^{g(g(x))} \frac{a'(u)}{u}\, du = \int_{a(s)}^{s} \frac{a'(u)}{u}\, du$$

for $s = g(g(x))$ and $x \in (0, a(a(r)))$. The result now follows from (22) of [3]. The second part of the lemma follows in analogous fashion from (20) of [3].

Finally, let $\psi\colon (-\infty, 0) \to (0, a(r))$ be the inverse of ϕ. Then $\psi \in C^2((-\infty, 0))$ with $\psi'(x) > \psi(x)$ and $\psi''(x) > 0$ for $x \in (-\infty, 0)$.

THEOREM 3. *The following equations hold*:

$$g(\psi(x)) = \psi'(x) \qquad \text{for} \quad x \in (-\infty, 0), \tag{4}$$

$$\psi'(\phi(x)) = g(x) \qquad \text{for} \quad x \in (0, a(r)), \tag{5}$$

$$\psi'(x) = \psi(x + c) \qquad \text{for} \quad x \in (-\infty, -c). \tag{**}$$

Proof. Since $\phi(\psi(x)) = x$ for $x \in (-\infty, 0)$, differentiation and (2) yield (4). Equation (5) is proved analogously. For $x \in (0, a(a(r)))$, (3) and (5) imply $\psi(\phi(x) + c) = g(x) = \psi'(\phi(x))$, so (**) holds for $x \in (-\infty, \phi(a(a(r))))$ by Lemma 2. But, letting $x \to a(a(r))$ in (3), we see that $\phi(a(a(r))) = -c$, completing the proof.

With an eye toward eventually reversing this series of substitutions, we state the following simple, inelegant result.

COROLLARY 4. *Define* $\psi(0) = a(r)$. *Then* $\psi \in C^1((-\infty, 0]) \cap C^2((-\infty, 0))$ *with*

(i) $\psi'(-c) = \psi(0)$,

(ii) $\psi''(-c) = \psi'(0)$,

(iii) $\psi' > \psi > 0$ *on* $[-c, 0]$,

(iv) $\psi'' > 0$ *on* $[-c, 0)$.

We also have $\psi'(0) = r$.

Proof. Property (i) follows from (**), (ii) follows by differentiating (**), and the last assertion of the corollary follows by letting $x \to 0$ in (4). Property (iii) then holds at zero by Lemma 1, and the remaining assertions have already been established.

We have assumed above that $r < \infty$. If $r = \infty$, then we fix r_1 in $(0, \infty)$ and redefine

$$\phi(x) = \int_{r_1}^{x} \frac{1}{g(t)}\, dt \qquad \text{for} \quad x \in (0, \infty). \tag{2$'$}$$

Equations (*) and (1) imply a, $g \in C^\infty((0, \infty))$, so $\phi \in C^\infty((0, \infty))$ also. The inequalities involving ϕ, Eq. (3), and Lemma 2 all remain valid; in particular, (3) implies $\lim_{x\to\infty} \phi(x) = \infty$. Again letting $\psi\colon (-\infty, \infty) \to (0, \infty)$ be the inverse of ϕ, we have $\psi \in C^\infty((-\infty, \infty))$ with $\psi'(x) > \psi(x)$ and

$\psi''(x) > 0$ for all real x; moreover, (5) remains valid, and (4) and (**) hold for all real x. Finally, properties (i) through (iv) of Corollary 4 clearly remain valid.

We now wish to explore the circumstances under which the above process is reversible. We make no assumptions about r.

THEOREM 5. *Fix $c \in (0, 1/e]$ and a real function $\psi_0 \in C^1([-c, 0]) \cap C^2([-c, 0))$ such that properties* (i) *through* (iv) *of Corollary* 4 *hold for ψ_0. By the method of steps, construct the unique function ψ such that $\psi'(x) = \psi(x+c)$ for $x \in (-\infty, -c]$ and $\psi \equiv \psi_0$ on $[-c, 0]$.* Suppose that $\psi > 0$ on $(-\infty, 0]$. *Continue ψ to the right as far as possible so that convexity is preserved, i.e., let b be the largest extended real number such that $\psi'(x) = \psi(x+c)$ for $x \in (-\infty, b-c)$, $\psi \in C^2((-\infty, b))$, and $\psi'' > 0$ on $(-\infty, b)$. Let $\psi(b)$ and $\psi'(b)$ be defined in the obvious fashion, with $\psi(b) = \psi'(b) = \infty$ if $b = \infty$. Finally, let $\phi\colon (0, \psi(b)) \to (-\infty, b)$ be ψ^{-1}, let $g = 1/\phi'$, and let $a\colon (0, \psi'(b)) \to (0, \psi(b))$ be g^{-1}. Then $a \in C^1((0, \psi'(b)))$ (if $b = \infty$, then $a \in C^\infty$) with $0 < a(x) < x$, $a'(x) > 0$, and $a'(a(x)) = a(x)/x$ for $x \in (0, \psi'(b))$.*

Outline of proof. The assumptions on ψ_0 and the fact that ψ is assumed to be a positive solution of (**) on $(-\infty, -c]$ guarantee that $\psi \in C^2((-\infty, 0))$ with $\psi'(x) > \psi(x) > 0$ and $\psi''(x) > 0$ for $x \in (-\infty, 0)$. (So $b \geqslant 0$.) Integrating (**) shows that $\lim_{x \to -\infty} \psi(x) = 0$. From the definition of b, it follows that $\psi'(x) > \psi(x) > 0$ for $x \in (-\infty, b)$. The remaining assertions follow easily by reversing our previous arguments.

THEOREM 6. *Assume all of the hypotheses of Theorem* 5. *Let $0 < \alpha < 1$ such that $c = -\alpha \log \alpha$, and let $r = \psi'(b)$. Suppose that the function $\exp(x/\alpha)/\psi''(x)$ is strictly monotone on $(-\infty, b)$ and approaches a finite limit as $x \to -\infty$. Then the function f defined by*

$$f(\psi'(x)) = K_0 \exp(x/\alpha)/\psi''(x) \qquad \text{for} \quad x \in (-\infty, b) \tag{6}$$

is an α-semiwombat function on $[0, r)$ for any $K_0 \neq 0$.

Proof. We wish to invoke [3, Theorem 14]. Conditions (i), (ii), and (iii) of that theorem hold by Theorem 5 above. Equations (20) and (22) of [3] are seen to hold by reversing the arguments in the proof of Lemma 2. Transforming Eq. (19) of [3] via our series of substitutions yields (6) in straightforward fashion, and the theorem follows.

We turn now to the key problem of generating positive solutions for (**), so that Theorems 5 and 6 will apply. In general, it is not clear how to specify ψ_0 in Theorem 5 so that its continuation to the left remains positive. In the next section, we illustrate a successful specification.

3. Some Solutions by the Method of Steps

Fix $0 < c \leqslant \frac{1}{4}$, and let $0 < \alpha < 1/e$ such that $c = -\alpha \log \alpha$. Let $\psi_0(x) = x^2 + 2x + 2 - 2c$ for $x \in [-c, 0]$. Properties (i) through (iv) of Corollary 4 clearly hold.

We proceed to construct ψ as in Theorem 5. By induction on n,

$$\psi(x) = \frac{2}{(n+2)!}(x+nc)^{n+2} + \frac{2}{(n+1)!}(x+nc)^{n+1} + \sum_{k=0}^{n} \frac{\psi(-kc)}{(n-k)!}(x+nc)^{n-k}$$

for $x \in [-(n+1)c, -nc]$, $n = 0, 1, 2, \ldots$. Hence

$$\psi(-(n+1)c) = \frac{2}{(n+2)!}(-c)^{n+2} + \frac{2}{(n+1)!}(-c)^{n+1} + \sum_{k=0}^{n} \frac{\psi(-kc)}{(n-k)!}(-c)^{n-k}.$$

To prove that $\psi > 0$ on $(-\infty, -c)$, it suffices to show that $\psi(-(n+1)c) > 0$ for all n. If we form the generating function $S(t) = \sum_{n=0}^{\infty} \psi(-nc)\, t^n$, an easy computation using the above recursion formula shows that $S(t) = (2 - 2e^{-ct} - 2te^{-ct})/(t^2 e^{-ct} - t)$ for $|t|$ small. Rearranging this and substituting t/c for t, we get

$$\frac{t}{2c} \cdot \sum_{n=0}^{\infty} \frac{\psi(-nc)}{c^n} t^n + 1 = 1/(e^t - t/c). \tag{7}$$

Theorem 7. $\psi(-(n+1)c) > \frac{1}{2}\psi(-nc) > 0$ *for all* n.

Proof (communicated by L. N. Bidwell). Let $h(t) = 1/(e^t - t/c)$ and let $\sum_{j=0}^{\infty} c_j t^j$ be the power series for $h(t)$ around $t = 0$. Since $ch(t) = te^{-t}h(t) + ce^{-t}$, term-by-term comparison of power series gives

$$c_{n+1} = 1/c(c_n - c_{n-1} + c_{n-2}/2! - c_{n-3}/3! + \cdots + (-1)^n c_0/n!) + (-1)^{n+1}/(n+1)!. \tag{8}$$

We shall inductively prove that $c_{n+1} > 1/2c \cdot c_n > 0$. The case $n = 0$ is obvious. Therefore we assume that $c_n > 1/2c \cdot c_{n-1}$ and that $c_n > c_{n-1} > c_{n-2} > \cdots > c_0 > 0$. From (8), we obtain

$$c_{n+1} \geqslant \frac{1}{c}(c_n - c_{n-1}) > \frac{1}{c}(c_n - 2c \cdot c_n) = \frac{1-2c}{c} \cdot c_n \geqslant \frac{1}{2c} \cdot c_n, \qquad \text{since} \quad c \leqslant \frac{1}{4}.$$

This completes the induction. Finally, (7) implies that $\psi(-nc) = 2c^{n+1}c_{n+1}$, and the result follows.

We have thus generated a positive solution to (**). By our definition of ψ_0, it is clear that $b = 0$ in Theorem 5. To apply Theorem 6 (with $r = \psi'(0) = 2$), it remains to discuss the monotonicity on $(-\infty, 0)$ and limit at $-\infty$ of the function $\exp(x/\alpha)/\psi''(x)$.

THEOREM 8. *For ψ and α as above, let $H(x) = \exp(x/\alpha)/\psi''(x)$ for $x \in (-\infty, 0)$. Then H is positive, strictly increasing, and* $\lim_{x\to-\infty} H(x) = 0$.

Proof. The positivity of H is apparent. Direct computation shows that H is strictly increasing on $[-2c, 0)$. For $x \leqslant -2c$, $H(x) = \exp(x/\alpha)/\psi(x+2c)$ so it will suffice to show that the function $G(x) = \exp(x/\alpha)/\psi(x)$ is strictly increasing on $(-\infty, 0]$. We shall show $G'(x) > 0$, that is, $\psi(x) - \alpha\psi'(x) > 0$ for all $x \leqslant 0$. We proceed by induction on n, where $x \in [-(n+1)c, -nc]$. For $n = 0$, direct computation gives the desired result. We therefore assume that $\psi(x) - \alpha\psi'(x) > 0$ for $x \in [-(n+1)c, -nc]$. Since $\alpha < \frac{1}{2}$, Theorem 7 implies that $\psi(x) - \alpha\psi'(x) > 0$ for $x = -(n+2)c$, and the induction hypothesis together with differentiation of (**) implies that $\psi(x) - \alpha\psi'(x)$ has positive first derivative on $[-(n+2)c, -(n+1)c]$. Hence $\psi(x) - \alpha\psi'(x) > 0$ for $x \in [-(n+2)c, -(n+1)c]$, completing the induction.

It remains to establish that $\lim_{x\to-\infty} H(x) = 0$. We already know that this limit exists and is non-negative, so Theorem 6 guarantees that the function $f(\psi'(x)) = K_0 H(x)$ for $x \in (-\infty, 0)$ is an α-semiwombat function on $[0, 2)$. Suppose this limit is non-zero. Then Lemma 10 of [3] implies $a'(0) = \alpha$. But Eq. (4) and Theorem 7 give $a'(0) = \lim_{x\to 0}(a(x)/x) = \lim_{x\to-\infty}(a(\psi'(x))/\psi'(x)) = \lim_{x\to-\infty}(\psi(x)/\psi(x+c)) = \lim_{n\to\infty}(\psi(-(n+1)c)/\psi(-nc)) \geqslant \frac{1}{2}$, whereas $\alpha < 1/e$. This contradiction completes the proof.

Open question: Can the arguments given in this section be sharpened to handle the case $\frac{1}{4} < c \leqslant 1/e$?

Remarks. The α-semiwombat functions obtained above can of course be "scaled" (see [3, p. 450].) In general, multiplying ψ by a positive constant has the effect of scaling a and f.

We also note that the resulting solutions a of (*) are not twice differentiable on $(0, r)$, resolving a question raised in [3, Sect. 5].

4. SOLUTIONS VIA EXPONENTIAL FUNCTIONS

In this section, we abandon the method of steps in favor of the usual technique of letting $\psi(x) = \sum_j q_j e^{\lambda_j x}$, where the λ_j's are (complex) roots of $e^{\lambda c} = \lambda$. We shall present our results in terms of finite sums, but they remain valid for infinite sums under the appropriate convergence assumptions.

For $0 < c < 1/e$, let λ_0, λ_1 denote the unique real solutions of $e^{\lambda c} = \lambda$, with $1 < \lambda_0 < e < \lambda_1$. For $c = 1/e$, let $\lambda_0 = \lambda_1 = e$ be the double real root of $e^{\lambda c} = \lambda$. In either case, if $\lambda \neq \lambda_0, \lambda_1$ is any other solution of $e^{\lambda c} = \lambda$, a standard argument gives $\operatorname{Re} \lambda > \lambda_1$.

THEOREM 9. *Fix $0 < c < 1/e$, and let $\lambda_j = \alpha_j + i\beta_j$, $j = 2,..., m$, be non-real roots of $e^{\lambda c} = \lambda$. Let*

$$\psi(x) = q_0 e^{\lambda_0 x} + q_1 e^{\lambda_1 x} + \sum_{j=2}^{m} (q_j e^{\alpha_j x} \cos \beta_j x + q_j' e^{\alpha_j x} \sin \beta_j x),$$

where q_0, q_1, q_j, q_j' are real constants. Consider the following two cases:

(A) *Let $\alpha = 1/\lambda_0$ and fix a real number b. Suppose that*

$$|q_1| \lambda_1^2(\lambda_1 - \lambda_0) > \sum_{j=2}^{m} \sqrt{q_j^2 + (q_j')^2}\, |\lambda_j|^2\, |\lambda_j - \lambda_0|\, e^{(\alpha_j - \lambda_1)b} \tag{9}$$

and

$$q_0 \lambda_0^2 \geqslant |q_1|\, \lambda_1^2\, e^{(\lambda_1 - \lambda_0)b} + \sum_{j=2}^{m} \sqrt{q_j^2 + (q_j')^2}\, |\lambda_j|^2\, e^{(\alpha_j - \lambda_0)b}. \tag{10}$$

Then $\psi(x)$ yields an α-semiwombat function through Eq. (6).

(B) *Let $\alpha = 1/\lambda_1$ and fix a real number b. Suppose that*

$$q_0 \lambda_0^2(\lambda_1 - \lambda_0) > \sum_{j=2}^{m} \sqrt{q_j^2 + (q_j')^2}\, |\lambda_j|^2\, |\lambda_j - \lambda_1|\, e^{(\alpha_j - \lambda_0)b} \tag{11}$$

and inequality (10) *holds. Then $\psi(x)$ yields an α-semiwombat function through Eq.* (6).

Proof. By linearity, it is clear that $\psi(x)$ satisfies (**) for all real x. Since $e^{\lambda_0 c} = \lambda_0$ and $e^{\lambda_1 c} = \lambda_1$, we see that $\alpha = 1/\lambda_0$ and $\alpha = 1/\lambda_1$ are the two solutions of $c = -\alpha \log \alpha$ in (0, 1). In case (A), condition (10) is sufficient to ensure that $\psi''(x) > 0$ on $(-\infty, b)$, and condition (9) ensures the monotonicity of $\exp(\lambda_0 x)/\psi''(x)$ on $(-\infty, b)$. Since this last function converges to $1/\lambda_0^2 q_0$ as $x \to -\infty$, the method of Theorems 5 and 6 applies. Case (B) is completely analogous.

Remark. It can easily be seen that the two simplest cases in Theorem 9, namely $\psi(x) = q_0 e^{\lambda_0 x}$ and $\psi(x) = q_0 e^{\lambda_0 x} + q_1 e^{\lambda_1 x}$, generate the collection of semiwombat functions found in [3]. (The parametric representation $\lambda_0 = (1 + 1/\beta)^\beta$, $\lambda_1 = (1 + 1/\beta)^{\beta+1}$, $0 < \beta < \infty$, is used in [3].)

THEOREM 10. *Let* $c = 1/e$, *and let* $\lambda_j = \alpha_j + i\beta_j$, $j = 2, \ldots, m$, *be non-real roots of* $e^{\lambda c} = \lambda$. *Let*

$$\psi(x) = q_0 e^{ex} + q_1 x e^{ex} + \sum_{j=2}^{m} (q_j e^{\alpha_j x} \cos \beta_j x + q_j' e^{\alpha_j x} \sin \beta_j x),$$

where q_0, q_1, q_j, q_j' *are real constants. Fix a real number* b. *Suppose that*

$$q_1 e^2 < -\sum_{j=2}^{m} \sqrt{q_j^2 + (q_j')^2}\, |\lambda_j|^2\, |\lambda_j - e|\, e^{(\alpha_j - e)b} \tag{12}$$

and

$$q_0 e^2 \geqslant |q_1|\, (e^2 b + 2e) + \sum_{j=2}^{m} \sqrt{q_j^2 + (q_j')^2}\, |\lambda_j|^2\, e^{(\alpha_j - e)b}. \tag{13}$$

Then $\psi(x)$ *yields a* $1/e$*-semiwombat through Eq.* (6).

Proof. Since $\lambda = e$ is a double root of $e^{\lambda/e} = \lambda$, it follows that xe^{ex} satisfies (**) for all real x, and that the only solution of $1/e = -\alpha \log \alpha$ is $\alpha = 1/e$. The rest of the proof parallels Theorem 9.

As a simple illustration of the above theorem, we take $\psi(x) = (1 - x)\, e^{ex}$ and $b = 1 - 2/e$. Then the function f: $[0, e^{e-2}) \to [0, \infty)$ given by $f(\psi'(x)) = 1/(e - 2 - ex)$ for $-\infty < x < 1 - 2/e$ is a $1/e$-semiwombat function on its domain. We remark that the resulting solution a of (*) is in $C^1([0, e^{e-2}))$ with $a'(0) = 1/e$ but it does not satisfy the hypothesis of [3, Theorem 15], resolving a previous question.

5. A Uniqueness Theorem and Some Historical Remarks

If we let $h(x) = \psi(-c(x-1))$, our basic equation $\psi'(x) = \psi(x + c)$ is transformed into the equation $h'(x+1) = -ch(x)$, which has been extensively studied by Schürer [5, 6]. We shall apply two of his results on the distribution of zeros of solutions to obtain the following complete description of the case $r = \infty$.

THEOREM 11. *Let* f *be an* α*-semiwombat function on* $[0, \infty)$. *Then one of the following holds*:

(1) $f(x) = qx^{\beta}$, *where* $\beta > 0$, $q \neq 0$.

(2) $f(x) = qf_{\beta,2}(kx)$, *where* $k > 0$, $q \neq 0$, *and* $f_{\beta,2}$ *is as in* [3, *Theorem* 17].

(3) $f(x) = q\tilde{f}_{\beta,2}(kx)$, *where* $k > 0$, $q \neq 0$, *and* $\tilde{f}_{\beta,2}$ *is as in* [3, *Theorem* 21].

Proof. By the discussion following Corollary 4, we know that f must give rise to a positive solution ψ of (**) for all real x. If $0 < c < 1/e$, [5, Theorem XIII] implies that $\psi(x) = q_0 e^{\lambda_0 x} + q_1 e^{\lambda_1 x}$, where λ_0, λ_1 are the real roots of $e^{\lambda c} = \lambda$, and $q_0 \geqslant 0$, $q_1 \geqslant 0$. The result now follows by the Remark following Theorem 9, and Theorems 16, 17, and 21 of [3]. If $c = 1/e$, [5, Theorem XVIII] implies that $\psi(x) = q_0 e^{ex}$, $q_0 > 0$, which implies $f \equiv$ constant. This contradiction completes the proof.

COROLLARY 12. *There exist a-semiwombat functions on* $[0, \infty)$ *if and only if* $a \neq 1/e$.

Proof. Immediate from the theorem.

Other results of Schürer [5, Theorems IIIb and XVIII] imply that all solutions of (**) are representable (on some smaller interval $(-\infty, d]$) as a uniformly absolutely convergent series of the form given in Theorems 9 or 10. Hence our approach in Section 4 is as general as possible.

We remark in closing that Eq. (1) was also studied by Pirondini [4]; Barba and Pirondini's interest in Eq. (1) stems from its connection with a geometry problem of Euler. See [5, p. 171] for further details and references.

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